LETTER TO THE EDITOR

Frequency Analysis Based on Easily Generated Functions

Yuchuan Wei [1]

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Abstract—In addition to the sine–cosine functions, sawtooth waves, square waves, triangular waves, trapezoidal waves, etc. are also easily generated periodic functions for modern electronics. Similar to Fourier analysis, a signal can be considered as a superposition of easily generated functions with different frequencies. In this letter, we discuss the change-of-bases formulas and consider the problem of convergence. Only the $L^2 := L^2[-\pi, \pi]$ setting is presented.

Fourier analysis has been playing an important role in signal analysis in that a signal is considered as a superposition of various sine and cosine functions with different frequencies. However, with the development of electronic techniques, sawtooth waves, square waves, triangular waves, and trapezoidal waves have become easily generated periodic functions. For example, it is quite convenient to obtain a system of square waves with different frequencies from a common high-frequency pulse by counters. A natural question is the following: *How well is a signal represented as a superposition of easily generated functions with different frequencies (or periods)?*

In mathematical terms, this problem is associated with completeness and convergence of the series expansions in terms of the general periodic functions.

From the viewpoint of practical applications, we have the following formal formulas for change of bases.

We suppose that $X(x)$ and $Y(x)$ are even and odd periodic functions with period 2π, with Fourier series

$$X(x) = \sum_{n=1}^{\infty} A(n) \cos nx \tag{1}$$

$$Y(x) = \sum_{n=1}^{\infty} B(n) \sin nx, \tag{2}$$

such that $A(1) \neq 0$ and $B(1) \neq 0$. Then it follows that

$$\cos x = \sum_{n=1}^{\infty} A^{-1}(n) X(nx) \tag{3}$$

[1] Department of Electrical Engineering, No. 202, Beijing University of Aeronautics and Astronautics, Beijing 100083, China. E-mail: yuchuawei@21cn.com.

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$$\sin x = \sum_{n=1}^{\infty} B^{-1}(n) Y(nx). \tag{4}$$

Here $A^{-1}(n)$ and $B^{-1}(n)$ denote the Dirichlet inverses of $A(n)$ and $B(n)$ (cf. [1, Chap. 2]).

Indeed,

$$\begin{aligned}\sum_{n=1}^{\infty} A^{-1}(n) X(nx) &= \sum_{n=1}^{\infty} A^{-1}(n) \sum_{m=1}^{\infty} A(m) \cos mnx \\ &= \sum_{k=1}^{\infty} \sum_{d|k} A(d) A^{-1}\left(\frac{k}{d}\right) \cos kx \\ &= \sum_{k=1}^{\infty} \delta_{k1} \cos kx \\ &= \cos x.\end{aligned}$$

Here δ is Kronecker's delta symbol. The justification for (4) is similar.

As usual, we write the Fourier series of a function $f(x)$ with period 2π as

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a(n) \cos nx + b(n) \sin nx), \tag{5}$$

where a_0, $a(n)$, $b(n)$ are the Fourier coefficients.

Putting (3) and (4) into (5), we obtain

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty} a(n) \sum_{l=1}^{\infty} A^{-1}(l) X(nlx) + \sum_{n=1}^{\infty} b(n) \sum_{l=1}^{\infty} B^{-1}(l) Y(nlx) \\ &= a_0 + \sum_{n=1}^{\infty} \left(\sum_{d|n} a(d) A^{-1}\left(\frac{n}{d}\right)\right) X(nx) + \sum_{n=1}^{\infty} \left(\sum_{d|n} b(d) B^{-1}\left(\frac{n}{d}\right)\right) Y(nx) \\ &= C_0 + \sum_{n=1}^{\infty} (C(n) X(nx) + D(n) Y(nx)), \end{aligned} \tag{6}$$

where the coefficients are

$$\begin{aligned} C_0 &= a_0 \\ C(n) &= \sum_{d|n} A^{-1}\left(\frac{n}{d}\right) a(d) = A^{-1} * a(n) \\ D(n) &= \sum_{d|n} B^{-1}\left(\frac{n}{d}\right) b(d) = B^{-1} * b(n). \end{aligned} \tag{7}$$

Here $*$ denotes multiplication of two arithmetical functions [1, Chap. 2].

These formal formulas may be enough for engineers to synthesize a signal or decompose a signal into easily generated functions with different frequencies, but the mathematical rigor has to be justified. In this letter, we only consider convergence in $L^2 := L^2[-\pi, \pi]$.

PROPOSITION 1. *If the Fourier coefficients of the functions* $X(x)$ *and* $Y(x)$ *are in* l^1 *and completely multiplicative* (*see* [1, Chap. 2]), *then the series* (3) *and* (4) *converge* *in L^2 and uniformly in $[-\pi, \pi]$. Furthermore, for any function $f(x) \in L^2$, the series* (6) *converges unconditionally to $f(x)$ in L^2.*

Proof. Since

$$\sum_{n=1}^{\infty} |A^{-1}(n)| \sum_{m=1}^{\infty} |A(m)\cos mnx| \le \sum_{n=1}^{\infty} |A(n)| \sum_{m=1}^{\infty} |A(m)| < \infty,$$

it follows by Weierstrass' M-test that the series (3) converges absolutely everywhere and uniformly in $[-\pi, \pi]$. Furthermore, the series (3) converges in L^2. The proof for (4) is similar.

Next, since

$$\begin{aligned}
&\left\| \sum_{k=1}^{\infty} \left(\sum_{nlm=k} |b(n)B^{-1}(l)B(m)| \right) \sin(kx) \right\| \\
&= \left\| \sum_{m=1}^{\infty} |B(m)| \sum_{l=1}^{\infty} |B^{-1}(l)| \sum_{n=1}^{\infty} |b(n)| \sin(nlmx) \right\| \\
&\le \sum_{m=1}^{\infty} |B(m)| \sum_{l=1}^{\infty} |B^{-1}(l)| \left\| \sum_{n=1}^{\infty} |b(n)| \sin(nlmx) \right\| \\
&\le \sum_{m=1}^{\infty} |B(m)| \sum_{l=1}^{\infty} |B(l)| \sqrt{\pi \sum_{n=1}^{\infty} b^2(n)} \\
&< \infty,
\end{aligned}$$

all of the series in

$$\begin{aligned}
\sum_{n=1}^{\infty} b(n)\sin(nx) &= \sum_{k=1}^{\infty} \left(\sum_{nlm=k} b(n)B^{-1}(l)B(m) \right) \sin(kx) \\
&= \sum_{n=1}^{\infty} b(n) \sum_{l=1}^{\infty} B^{-1}(l) \sum_{m=1}^{\infty} B(m)\sin(nlmx) \\
&= \sum_{n=1}^{\infty} b(n) \sum_{l=1}^{\infty} B^{-1}(l) Y(nlx) \\
&= \sum_{n=1}^{\infty} D(n) Y(nx)
\end{aligned}$$

are convergent in L^2.

For the same reason, the series

$$\sum_{n=1}^{\infty} a(n)\cos(nx) = \sum_{n=1}^{\infty} a(n) \sum_{l=1}^{\infty} A^{-1}(l) X(nlx) = \sum_{n=1}^{\infty} C(n) X(nx)$$

converges unconditionally as well. This completes the proof. ■

In fact, in this case the function system (1), $\{X(nx), Y(nx)\}_{n=1}^{\infty}$, is an unconditional basis of L^2.

Two examples of practical importance are the even and odd triangular waves (see [2])

$$X_{\text{tri}}(x) = \begin{cases} \frac{\pi}{4}(x + \frac{\pi}{2}) & x \in [-\pi, 0) \\ \frac{\pi}{4}(-x + \frac{\pi}{2}) & x \in [0, \pi) \end{cases}$$

$$= \cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \frac{1}{7^2}\cos 7x + \cdots$$

$$Y_{\text{tri}}(x) = \begin{cases} \frac{\pi}{4}x & -\frac{\pi}{2} \le x < \frac{\pi}{2} \\ \frac{\pi^2 - \pi x}{4} & \frac{\pi}{2} \le x < \frac{3\pi}{2} \end{cases}$$

$$= \sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \frac{1}{7^2}\sin 7x + \cdots,$$

and the even and odd trapezoidal waves (see [2])

$$X_{\text{tra}}(x) = \begin{cases} -\frac{\sqrt{2\pi^2}}{16} & x \in [-\frac{5\pi}{4}, -\frac{3\pi}{4}] \\ \frac{\sqrt{2\pi}}{4}(x + \frac{\pi}{2}) & x \in [-\frac{3\pi}{4}, \frac{-\pi}{4}] \\ \frac{\sqrt{2\pi^2}}{16} & x \in [\frac{-\pi}{4}, \frac{\pi}{4}] \\ \frac{\sqrt{2\pi}}{4}(\frac{\pi}{2} - x) & x \in [\frac{\pi}{4}, \frac{3\pi}{4}] \end{cases}$$

$$= \cos x - \frac{1}{3^2}\cos 3x - \frac{1}{5^2}\cos 5x + \frac{1}{7^2}\cos 7x + \cdots$$

$$Y_{\text{tra}}(x) = \begin{cases} -\frac{\sqrt{2\pi^2}}{16} & x \in [-\frac{3\pi}{4}, -\frac{\pi}{4}] \\ \frac{\sqrt{2\pi}}{4}x & x \in [-\frac{\pi}{4}, \frac{\pi}{4}] \\ \frac{\sqrt{2\pi^2}}{16} & x \in [\frac{\pi}{4}, \frac{3\pi}{4}] \\ \frac{\sqrt{2\pi}}{4}(\pi - x) & x \in [\frac{3\pi}{4}, \frac{5\pi}{4}] \end{cases}$$

$$= \sin x + \frac{1}{3^2}\sin 3x - \frac{1}{5^2}\sin 5x - \frac{1}{7^2}\sin 7x + \cdots.$$

We also have the following.

PROPOSITION 2. *If the Fourier coefficients $A(n)$ and $B(n)$ of $X(x)$, $Y(x) \in L^2$ are completely multiplicative, then the series* (3) *and* (4) *converge unconditionally in L^2. Furthermore, for any function $f(x) \in L^2$ whose Fourier coefficients satisfy*

$$\sum_{n=1}^{\infty} a^2(n)d(n) < \infty \qquad \textit{and} \qquad \sum_{n=1}^{\infty} b^2(n)d(n) < \infty,$$

the series (6) *converges to $f(x)$ in L^2. Here $d(n)$ denotes the number of divisors of positive integer n.*

Proof. The proof is similar to the proof in [5], except for the use of the inequalities

$$\sum_{n=1}^{\infty} d^2(n)A^2(n) \le \left(\sum_{n=1}^{\infty} A^2(n)\right)^4 < \infty \tag{8}$$

and

$$\sum_{j=1}^{\infty} B^2(j)d^3(j) \le \left(\sum_{j=1}^{\infty} B^2(j)\right)^8 < \infty. \tag{9}$$

To verify (8) and (9), we use the property of the divisor function, $d(m)d(n) \geq d(mn)$, and note that

$$d * d(n) = \sum_{m|n} d(m) d\left(\frac{n}{m}\right) \geq \sum_{m|n} d(n) = d^2(n),$$

so that

$$d * d * d * d(n) \geq d^2 * d^2(n) \geq d^3(n).$$

Therefore the inequalities (8) and (9) hold. ■

In fact, in this case the function system (1), $\{X(nx), Y(nx)\}_{n=1}^{\infty}$, is a basis of the subspace

$$K = \{f(x) \in L^2[-\pi, \pi] : f(x) = a_0 + \textstyle\sum_{n=1}^{\infty} a(n) \cos nx + b(n) \sin nx \\ \text{with } \textstyle\sum_{n=1}^{\infty} a^2(n) d(n) < \infty \text{ and } \textstyle\sum_{n=1}^{\infty} b^2(n) d(n) < \infty\}$$

of L^2, but whether it is a basis of L^2 is still not clear to us (see [5]).

Two examples of practical importance are sawtooth waves (see [2])

$$\begin{aligned} Y_{\text{sa}}(x) &= \begin{cases} (\pi - x)/2 & 0 < x < 2\pi \\ 0 & x = 0 \end{cases} \\ &= \sin x + \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) + \cdots + \frac{1}{n}\sin(nx) + \cdots, \end{aligned}$$

and the even and odd square waves (see [2, 3, 5])

$$\begin{aligned} X_{\text{sq}}(x) &= \frac{\pi}{4}\operatorname{sgn}(\cos x) \\ &= \cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \cdots \\ Y_{\text{sq}}(x) &= \frac{\pi}{4}\operatorname{sgn}(\sin x) \\ &= \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \cdots. \end{aligned}$$

Here sgn denotes the signum function; i.e.,

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0. \end{cases}$$

From the viewpoint of applications, many techniques based on sine–cosine functions such as filtering can be translated into techniques based on easily generated functions. The reader is referred to [4–7] for further discussions.

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